Quiz 2 **December 12th, 2012**

Dynamic Programming & Optimal Control (151-0563-00)

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Solutions

Duration: **45 minutes**

Number of Problems: **3**

Permitted Aids: Provided cheat sheet on page 1.

Use only the prepared sheets for your solutions.

Cheat sheet

Hamilton-Jacobi-Bellman Equation

$$0 = \min_{u \in U} \left[g(x,u) + \frac{\partial V(t,x)}{\partial t} + \left(\frac{\partial V(t,x)}{\partial x} \right)^{\mathrm{T}} f(x,u) \right], \quad \text{for all } x \text{ and } t \in [0,T],$$

$$V(T,x) = h(x), \quad \text{for all } x.$$

Pontryagin's Minimum Principle

Hamiltonian:

$$H\left(x,u,p\right) = g\left(x,u\right) + p^{\mathrm{T}} f\left(x,u\right)$$

Adjoint equations:

$$\dot{p} = -\frac{\partial H\left(x^*,u^*,p\right)}{\partial x}$$

Problem 1 **20%**

Given is the general continuous-time optimal control problem as defined in class with dynamics

$$\dot{x}(t) = f(x(t), u(t)), \quad 0 \leq t \leq T, \quad x(0) : \text{given}. \tag{1}$$

The objective of the continuous-time optimal control problem is to minimize a cost function of the form

$$h(x(T)) + \int_0^T g(x(t), u(t))dt. \tag{2}$$

Which of the following statements are true? Tick the correct answer.

a) If a candidate cost-to-go function $J(t, x)$ satisfies the Hamilton-Jacobi-Bellman equations the corresponding minimizing policy $\mu(t, x)$ is an optimal policy.

√ ○ True

○ False

b) All input trajectories that satisfy the Minimum Principle equations are optimal.

○ True

√ ○ False

c) If an input $u(t)$ and the corresponding state trajectory $x(t)$ satisfy the Hamilton-Jacobi-Bellman equations (under a certain cost-to-go function), then $u(t)$ and $x(t)$ also satisfy the Minimum Principle equations.

√ ○ True

○ False

d) The optimal cost-to-go function in the case of a free initial state is smaller than or equal to the optimal cost-to-go function with fixed initial state.

√ ○ True

○ False

Problem 2 **40%**

Consider the one-dimensional linear system

$$\dot{x}(t) = ax(t) + bu(t), \quad x(0) = x_0 \in \mathbb{R}, \quad t \in [0, T],$$

where $u \in \mathbb{R}$ is the input, T is given, and $a, b \in \mathbb{R}$. The objective is to find an optimal policy that minimizes the quadratic cost

$$x^2(T) + \int_0^T (x^2(t) + u^2(t))dt.$$

Show that the cost-to-go function given by

$$J(t, x) = k(t)x^2,$$

where $k(t) \in \mathbb{R}$ is the solution to

$$\dot{k}(t) = -2a\ k(t) + b^2 k^2(t) - 1, \quad k(T) = 1, \quad t \in [0, T],$$

is the *optimal* cost-to-go function and find the corresponding optimal policy as a function of $k(t)$ and x.

Solution 2

Let

$$G(t, x) := \min_{u \in U} \left[g(x, u) + \nabla_t V(t, x) + \nabla_x V^{\mathrm{T}}(t, x) f(x, u) \right]$$

denote the right hand side of the Hamilton-Jacobi-Bellman equation. Now substituting $g(x, u) = x^2 + u^2$ (stage cost) and $V(t, x) = J(t, x) = k(t)x^2$ (optimal cost-to-go function candidate) gives

$$G(t, x) = \min_{u \in U} \left[x^2 + u^2 + \dot{k}(t)x^2 + 2k(t)x(ax + bu) \right]. \tag{3}$$

The minimum is attained at a u for which the gradient with respect to u is zero; that is,

$$2u + 2bk(t)x = 0$$

or

$$u = -bk(t)x.$$

Since the second derivative w.r.t. u is bigger than zero, this is the minimum.
Substituting the minimizing value of u in (3) gives

$$\begin{aligned} G(t, x) &= x^2 + b^2 k^2(t) x^2 + \dot{k}(t) x^2 + 2ak(t)x^2 - 2b^2 k^2(t) x^2 \\ &= x^2(\dot{k}(t) + 2ak(t) - b^2 k^2(t) + 1) \\ &= 0 \quad \forall x, t \in [0, T]. \end{aligned}$$

Note that the last equality results from the fact that $k(t)$ is a solution to

$$\dot{k}(t) = -2ak(t) + b^2 k^2(t) - 1.$$

Furthermore,

$$V(T, x) = J(T, x) = k(T)x^2 = x^2 = h(x)(\text{terminal cost}), \quad \forall x.$$

Therefore, $J(t, x)$ is the optimal cost-to-go, i.e., $J^*(t, x) = J(t, x)$ and the corresponding optimal policy is given by

$$\mu^*(t, x) = -bk(t)x.$$

Problem 3 40%

Consider the predator-prey model that is given by the following dynamic system:

$$\begin{aligned}\dot{x}(t) &= x(t)\left(\alpha - \beta y(t) - u_1(t)\right)\\ \dot{y}(t) &= -y(t)\left(\gamma - \delta x(t) + u_2(t)\right),\end{aligned}$$

where $x(t)$ represents the population of the prey and $y(t)$ the population of predators with given initial states $x(0) = x_0 > 0$, $y(0) = y_0 > 0$. The growth rate of the prey is described by $\dot{x}(t)$ and the growth rate of the predators by $\dot{y}(t)$. The hunters can influence the populations by $u_1, u_2 \in [0, 1]$. We consider the time horizon $t \in [0, T]$ with given terminal time $T \in \mathbb{R}^+$. The parameter $\alpha > 0$ describes the reproduction rate of the prey, and the parameter $\beta > 0$ describes the predation rate. The parameter $\gamma > 0$ represents the death rate of the predators, and the parameter $\delta > 0$ is the reproduction rate of the predators. This model assumes that there is unlimited food available for the prey and that the predators' only food supply is the prey population. The parameters α, β, γ, δ are constants. It can be assumed that $x(t), y(t) > 0$ for all $t \in [0, T]$.
The influence of the hunters, u_1 and u_2, should be used to keep both populations at around constant levels c_1 and c_2. That is, the goal is to minimize the following cost function

$$\int_0^T \left((x(t) - c_1)^2 + (y(t) - c_2)^2\right) dt.$$

a) Write down the Hamiltonian function for the given problem.

b) Derive the adjoint equations and their boundary values for the given problem. (You do not need to solve them!)

c) Find the optimal inputs $u_1^*(t)$ and $u_2^*(t)$ as a function of $p(t)$, $x^*(t)$, and $y^*(t)$. (You can ignore the instances in time where $p(t) = 0$.)

d) How would the solution of a), b), c) change if we require the terminal state to be $x(T) = c_1$, $y(T) = c_2$?

Solution 3

a) Hamiltonian: (using $\mathbf{x} = [x, y]^{\mathrm{T}}$, $u = [u_1, u_2]^{\mathrm{T}}$, and $p = [p_1, p_2]^{\mathrm{T}}$)

$$H(\mathbf{x}, u, p) = g(\mathbf{x}, u) + p^T f(\mathbf{x}, u) = (x - c_1)^2 + (y - c_2)^2 + p_1 x (\alpha - \beta y - u_1) - p_2 y (\gamma - \delta x + u_2)$$

b) Adjoint equations:

$$\begin{aligned}
\dot{p_1} &= -\frac{\partial H}{\partial x} = -2(x - c_1) - p_1(\alpha - \beta y - u_1) - \delta p_2 y \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad p_1(T) = 0 \\
\dot{p_2} &= -\frac{\partial H}{\partial y} = -2(y - c_2) + p_2(\gamma - \delta x + u_2) + \beta p_1 x \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad p_2(T) = 0
\end{aligned}$$

c) Optimal inputs:
If $u_1^*(t)$ and $u_2^*(t)$ are the optimal inputs and $\mathbf{x}^*(t) = [x^*(t), y^*(t)]^{\mathrm{T}})$ is the optimal state trajectory, then the necessary condition for optimality is

$$\begin{aligned}
u^*(t) = [u_1^*(t), u_2^*(t)]^{\mathrm{T}} &= \underset{0 \le u_1, u_2 \le 1}{\operatorname{argmin}} \; H(\mathbf{x}^*(t), u(t), p(t)) \\
&= \underset{0 \le u_1, u_2 \le 1}{\operatorname{argmin}} \left[(x(t) - c_1)^2 + (y(t) - c_2)^2 + p_1(t) x(t) (\alpha - \beta y(t) - u_1(t)) \right. \\
&\qquad\qquad \left. - p_2(t) y(t) (\gamma - \delta x(t) + u_2(t)) \right].
\end{aligned}$$

Since the Hamiltonian is linear in u_1 and u_2, the optimal inputs are on the boundaries:

$$u_1^*(t) = \begin{cases} 0 \text{ if } p_1(t) x^*(t) < 0 \\ 1 \text{ if } p_1(t) x^*(t) > 0 \end{cases} \quad u_2^*(t) = \begin{cases} 0 \text{ if } p_2(t) y^*(t) < 0 \\ 1 \text{ if } p_2(t) y^*(t) > 0 \end{cases}$$

Since we ignored the analysis at $p_1(t) = 0$ and $p_2(t) = 0$ and assumed $x(t), y(t) > 0$ for $t \in [0, T]$, the cases $p_1(t) x^*(t) = 0$ and $p_2(t) y^*(t) = 0$ can be neglected.

d) The Hamiltonian in a) and the optimal inputs in c) will not change. However, if the terminal state is fixed, the boundary conditions are defined purely on x: $x(0) = x_0, y(0) = y_0, x(T) = c_1, y(T) = c_2$; there is no terminal condition on the adjoint equation anymore. That is $p_1(T)$ and $p_2(T)$ are free.