Solutions

Duration: 45 minutes

Number of Problems: 1

Permitted Aids: None.

Use only the prepared sheets for your solutions.
Problem 1

Consider the following dynamical system,

\[ \dot{x}(t) = u(t), \quad 0 \leq t \leq T, \quad x(0) = x_0 \]

where \( x(t) \in \mathbb{R} \) and \( u(t) \in \mathbb{R} \). \( x_0 \) and \( T \) are fixed and given.

a) Calculate the optimal trajectory \( x^*(t) \) and optimal control input \( u^*(t) \) that minimize

\[ J = \frac{1}{2} \int_0^T (x^2(t) + u^2(t)) \, dt. \]

b) Find \( x^*(t) \) and \( u^*(t) \) as \( T \to \infty \). Furthermore, calculate the optimal cost

\[ J^*_\infty = \lim_{T \to \infty} \frac{1}{2} \int_0^T (x^*^2(t) + u^*^2(t)) \, dt. \]

c) Find a solution \( V(t, x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R} \) to the following partial differential equation

\[ 0 = \min_u \left( \frac{1}{2} (x^2 + u^2) + \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x} u \right), \quad t \geq 0. \]
Solution 1

a) Apply the minimum principle:

- The Hamiltonian is given by
  \[ H(x, u, p) = \frac{1}{2} (x^2 + u^2) + pu \]

- The adjoint equations follow from the equation above
  \[ \dot{p}(t) = -\frac{\partial H}{\partial x} = -x(t), \quad p(T) = 0 \] (no terminal cost)

- The optimal input is obtained by minimizing the Hamiltonian along the optimal trajectory
  \[ \frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u + p = 0 \quad \Rightarrow \quad u = -p \]

- Now \( \dot{x} = -p \) and \( \dot{p} = -x \) yield
  \[ \ddot{x} = x, \quad x(0) = x_0, \quad \dot{x}(T) = 0. \]
  
- Solving the above differential equation gives
  
  **Method 1:** Candidate solution \( x(t) = A \cosh(t) + B \sinh(t) \)
  Using initial conditions \( x(0) = x_0, \) \( \dot{x}(T) = 0 \) and \( \dot{x} = A \sinh(t) + B \cosh(t), \)
  we get \( A = x_0 \) and \( B = -x_0 \frac{\sinh(T)}{\cosh(T)}. \) This gives,
  \[ x(t) = x_0 \cosh(t) - x_0 \frac{\sinh(T)}{\cosh(T)} \sinh(t) \]
  \[ u(t) = \dot{x}(t) = x_0 \sinh(t) - x_0 \frac{\sinh(T)}{\cosh(T)} \cosh(t) \]

  **Method 2:** Candidate solution \( x(t) = A'e^t + B'e^{-t} \)
  Using initial conditions \( x(0) = x_0, \) \( \dot{x}(T) = 0 \) and \( \dot{x} = A'e^t - B'e^{-t}, \)
  we get \( A' = \frac{x_0}{1+e^{2T}} \) and \( B' = \frac{-x_0}{1+e^{-2T}}. \) This gives,
  \[ x(t) = \frac{x_0}{1+e^{2t}} e^t + \frac{x_0}{1+e^{-2t}} e^{-t} \]
  \[ u(t) = \dot{x}(t) = \frac{x_0}{1+e^{2t}} e^t - \frac{x_0}{1+e^{-2t}} e^{-t} \]

b) Optimal solution for infinite horizon setting:

- Using the solution of Method 1 in a)
  \[ x(t) = x_0 \left( \frac{e^t + e^{-t}}{2} - \left( \frac{e^T - e^{-T}}{e^t + e^{-t}} \right) \left( \frac{e^t - e^{-t}}{2} \right) \right) \]
  as \( T \to \infty, \) \( x(t) \to x_0 e^{-t} \)
  similarly, \( u(t) \to -x_0 e^{-t} \)
  \[ J^*_\infty = \frac{1}{2} \int_0^\infty \left( x_0^2 e^{-2t} + x_0^2 e^{-2t} \right) dt = \frac{x_0^2}{2} e^{-2t} \bigg|_0^\infty = \frac{x_0^2}{2} \]
\( J^*(t, x) = \frac{1}{2} \int_0^T \left( x^2(t) + u^2(t) \right) dt. \)
\[ = \frac{x_0^2}{2} \frac{1 - e^{-4T}}{(1 + e^{-2T})^2} \]
\[ \Rightarrow J^*_\infty(t, x) = \lim_{T \to \infty} J^*(t, x) = \frac{x_0^2}{2} \]

c) This is the Hamilton-Jacobi-Bellman equation for the above optimal control problem. For the above problem, if we find ourselves at state \( x \) at time \( t \), the optimal cost to go is \( \frac{x^2}{2} \). Therefore, \( V(t, x) = \frac{x^2}{2} \) is a candidate solution.

Verify:

\[ \frac{\partial V}{\partial t} = 0, \quad \frac{\partial V}{\partial x} = x \]
\[ \Rightarrow \min_u \left( \frac{1}{2} (x^2 + u^2) + xu \right) \]
occurs when \( u = -x \). Then we have \( \frac{1}{2} (x^2 + x^2) - x^2 = 0 \), as required.