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Quiz 1 | **October 13th, 2010**

Dynamic Programming & Optimal Control (151-0563-00)

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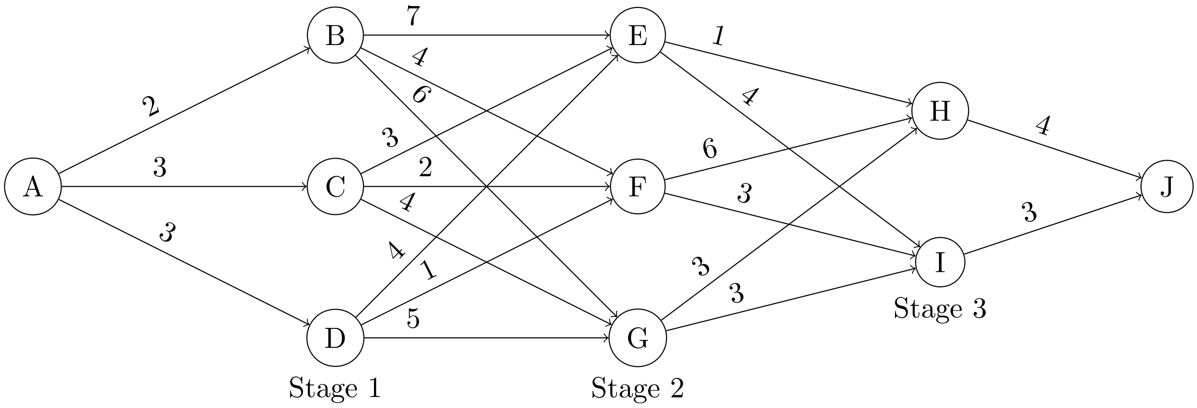
Solutions

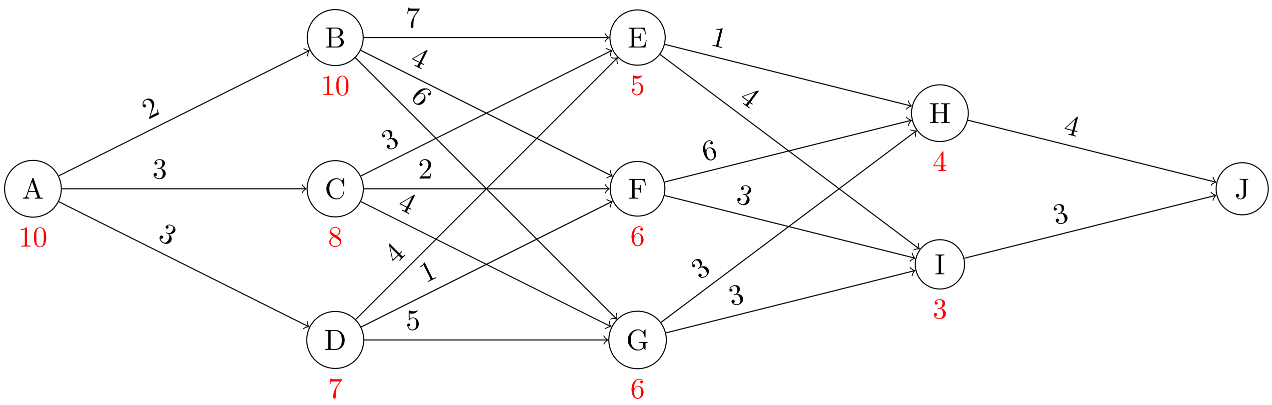
Duration:	**45 minutes**
Number of Problems:	**3**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 20%

The following graph describes a flexible production line. The raw product A has to through three stages to become the final product J. The cost for going from one node to another is stated next to the corresponding edge.
Calculate the optimal cost-to-go for each node and state the steps that should be taken to produce at minimal costs.



Solution 1



<u>Stage 4:</u>
$J_4(J) = 0$

<u>Stage 3:</u>
$J_3(H) = 4$
$J_3(I) = 3$

<u>Stage 2:</u>
$J_2(E) = min\,\{1 + J_3(H), 4 + J_3(I)\} = 5$
$J_2(F) = min\,\{6 + J_3(H), 3 + J_3(I)\} = 6$
$J_2(G) = min\,\{3 + J_3(H), 3 + J_3(I)\} = 6$

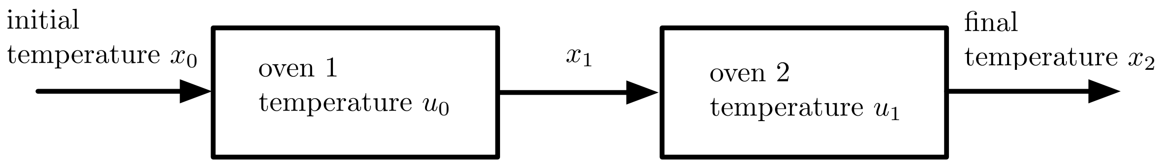
<u>Stage 1:</u>
$J_1(B) = min\,\{7 + J_2(E), 4 + J_2(F), 6 + J_2(G)\} = 10$
$J_1(C) = min\,\{3 + J_2(E), 2 + J_2(F), 4 + J_2(G)\} = 8$
$J_1(D) = min\,\{4 + J_2(E), 1 + J_2(F), 5 + J_2(G)\} = 7$

<u>Stage 0:</u>
$J_0(A) = min\,\{2 + J_1(B), 3 + J_1(C), 3 + J_1(D)\} = 10$

Optimal production steps: A, D, F, I, J

Problem 2 40%

A certain material is passed through a sequence of two ovens as shown in the following figure.



x_0 is the initial temperature of the material, x_k, $k = 1, 2$, is the temperature of the material at the exit of oven k, and u_{k-1}, $k = 1, 2$, is the temperature of oven k. The system dynamics is given by

$$x_{k+1} = x_k + u_k.$$

a) Find the optimal policy $u_k = \mu_k(x_k)$ that minimizes the cost given by $(T - x_2)^2 + u_1^2 + u_0^2$, where T is the desired final temperature.

b) Calculate the final temperature of the material, given $x_0 = t_0$, under the optimal policy.

c) State how the cost function can be modified in order for the final temperature to be closer to the desired final temperature.

Solution 2

a) $N = 2$ and the terminal cost is

$$g_2(x_2) = (T - x_2)^2 = J_2(x_2).$$

For $k = 1$ we have,

$$\begin{aligned} J_1(x_1) &= \min_{u_1}[u_1^2 + J_2(x_2)] \\ &= \min_{u_1}[u_1^2 + J_2(x_1 + u_1)] \\ &= \min_{u_1}[u_1^2 + (T - x_1 - u_1)^2]. \end{aligned} \tag{1}$$

Since the second derivative of the above cost w.r.t. u_1 is greater than zero, the cost can be minimized by setting its derivative w.r.t. u_1, to zero. This gives,

$$\begin{aligned} 0 &= 2u_1 - 2(T - x_1 - u_1), \\ \Longrightarrow \quad u_1 &= \frac{T - x_1}{2} = \mu_1(x_1). \end{aligned}$$

Finally, by substituting the above optimal u_1 on (1) gives,

$$J_1(x_1) = \frac{(T - x_1)^2}{2}.$$

Now, a similar calculation as above for $k = 0$ gives,

$$\begin{aligned} u_0 &= \frac{T - x_0}{3} = \mu_0(x_0), \\ J_0 &= \frac{(T - x_0)^2}{3}. \end{aligned}$$

b) Setting $x_0 = t_0$,

$$\begin{aligned} x_1 &= x_0 + u_0 = t_0 + \frac{T - t_0}{3} = \frac{T + 2t_0}{3} \\ &= \frac{T + 2t_0}{3}, \\ \Longrightarrow \text{final temperature} \quad x_2 &= x_1 + u_1 = x_1 + \frac{T - x_1}{2} = \frac{T + x_1}{2} \\ &= \frac{T + (\frac{T+2t_0}{3})}{2} = \frac{2T + t_0}{3}. \end{aligned}$$

c) Giving a higher weight to the cost that penalizes the deviation from the final temperature causes the final temperature x_2 to get closer to the desired final temperature T.
Example: $(T - x_2)^2 + u_1^2 + u_0^2 \quad \longrightarrow \quad r(T - x_2)^2 + u_1^2 + u_0^2,\ r > 1\ (r \to \infty \Rightarrow x_2 \to T)$

Alternatively you could avoid penalizing your input in the cost function. That would lead to the desired temperature, too. Example: $(T - x_2)^2 + u_1^2 + u_0^2 \quad \longrightarrow \quad (T - x_2)^2$

Problem 3 40%

Consider the dynamic system

$$x_{k+1} = x_k + u_k + w_k, \qquad k = 0, 1, 2$$

with initial state $x_0 = 0$, the state constraints $-2 \leq x_k \leq 4$ for all k and the disturbance

$$w_k = \begin{cases} 1 & \text{with probability } \frac{1}{2} \\ -1 & \text{with probability } \frac{1}{2} \end{cases} \quad \text{for all } k.$$

The cost incurred at time $k = 0, 1, 2$ is given by

$$g_k(x_k, u_k) = \begin{cases} x_k^2 + u_k^2 & 0 \leq x_k \leq 2 \\ x_k^2 + u_k^2 + 13 & x_k < 0 \text{ or } x_k > 2. \end{cases}$$

Given the terminal cost $g_3(x_3) = 0$, minimize the cost function

$$\sum_{k=0}^{2} g_k(x_k, u_k)$$

and calculate the optimal policy $u_k = \mu_k(x_k)$ under the following input constraints

$$u_k \in \begin{cases} \{1\} & \text{if } x_k < 0 \\ \{-1\} & \text{if } x_k > 2 \\ \{-1, 0, 1\} & x_k = 0, 1, 2. \end{cases}$$

Solution 3

<u>$k = 3$:</u>

$$g_3(x_3) = 0 = J_3(x_3)$$

<u>$k = 2$:</u>

$$J_2(x_2) = \min_{u_2} E_{w_2}\left[g_2(x_2, u_2) + \underbrace{J_3(x_3)}_{=0}\right]$$

$x_2 \backslash u_2$	-1	0	1
-2	-	-	18
-1	-	-	15
0	1	0	1
1	2	1	2
2	5	4	5
3	23	-	-
4	30	-	-

$\Longrightarrow$

x_2	$J_2(x_2)$	$\mu_2(x_2)$
-2	18	1
-1	15	1
0	0	0
1	1	0
2	4	0
3	23	-1
4	30	-1

<u>$k = 1$:</u>

$$\begin{aligned} J_1(x_1) &= \min_{u_1} E_{w_1}\left[g_1(x_1, u_1) + J_2(x_2)\right] \\ &= \min_{u_1}\left[g_1(x_1, u_1) + \frac{1}{2}J_2(x_1 + u_1 + 1) + \frac{1}{2}J_2(x_1 + u_1 - 1)\right] \end{aligned}$$

$x_1 \backslash u_1$	-1	0	1
-2	-	-	27
-1	-	-	23
0	10	8	3
1	10	3	14
2	7	16	23
3*	35	-	-
4*	47	-	-

$\Longrightarrow$

x_1	$J_1(x_1)$	$\mu_1(x_1)$
-2	27	1
-1	23	1
0	3	1
1	3	0
2	7	-1
3*	35	-1
4*	47	-1

*No need to calculate these states because they can't be reached from $x_0 = 0$.

<u>$k = 0$:</u>

$$\begin{aligned} J_0(x_0) &= \min_{u_0} E_{w_0}\left[g_0(x_0, u_0) + J_1(x_1)\right] \\ &= \min_{u_0}\left[g_0(x_0, u_0) + \frac{1}{2}J_1(x_0 + u_0 + 1) + \frac{1}{2}J_1(x_0 + u_0 - 1)\right] \end{aligned}$$

$x_1 \backslash u_1$	-1	0	1
0	16	13	6

$\Longrightarrow$

x_1	$J_1(x_1)$	$\mu_1(x_1)$
0	6	1