Dynamic Programming and Optimal Control
Recitation #2

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\[
J_N(x_N) = g_N(x_N)
\]
\[
J_K(x_k) = \min_{u_k \in U_k(x_k)} \mathbb{E} \left\{ \sum_{\omega_k} g_k(x_k, u_k, \omega_k) + J_{K+1}(f_k(x_k, u_k, \omega_k)) \right\}
\]
Refresher: Proof by Induction

Theorem

\[ 1 + 2 + \ldots + n = \frac{n(n+1)}{2}, \quad n \in \mathbb{N} \]

Proof.

- True for \( n = 1 \)
- Assume \( n = k \) holds: \( 1 + 2 + \ldots + k = \frac{k(k+1)}{2} \) (Induction Hypothesis)
- Show \( n = k + 1 \) holds:

\[
1 + 2 + \ldots + k + (k + 1) = \frac{k(k+1)}{2} + (k + 1)
\]

(by the Induction Hypothesis)

\[
= (k(k + 1) + 2(k + 1))/2
\]

\[
= (k + 2)(k + 1)/2
\]

\[
= (k + 1)(k + 2)/2
\]
1. Revising proof by induction.
2. Revising Dynamic Programming Algorithm.
3. Exercise 1.3 pg 52 (solution in problem set)
   [for a similar question see Problem 2, Quiz 1, 2011]

1.3

State space \( S_k = \{ R, B \} \)
- \( R \) - machine running
- \( B \) - machine broken

Control space \( U_k(x_k) \)
- \( u_k(R) = \{ n, m \} \)
  - \( n \) - no maintenance
  - \( m \) - maintenance
- \( u_k(B) = \{ r, l \} \)
  - \( r \) - repair
  - \( l \) - replace

System dynamics
\[ x_{k+1} = \omega_k \]

\[\begin{align*}
P(\omega_k = B \mid x_k = R, u_k = m) &= 0.4 \\
P(\omega_k = R \mid x_k = R, u_k = m) &= 1 - 0.4 = 0.6 \\
P(\omega_k = B \mid x_k = R, u_k = n) &= 0.7 \\
P(\omega_k = R \mid x_k = R, u_k = n) &= 1 - 0.7 = 0.3 \\
P(\omega_k = B \mid x_k = B, u_k = r) &= 0.4 \\
P(\omega_k = R \mid x_k = B, u_k = r) &= 1 - 0.4 = 0.6 \\
P(\omega_k = R \mid x_k = B, u_k = l) &= 1 \\
P(\omega_k = B \mid x_k = B, u_k = l) &= 1 - 1 = 0
\end{align*}\]

Stage cost = input cost \( C_w \) + Gain \( g_k \)

\[\begin{align*}
C_n &= 0 \\
C_m &= 20 (\$) \\
C_r &= 40 \\
C_l &= 150
\end{align*}\]

Gain function \( g_k \)
\[ g_k = \begin{cases} 
-6 & \text{if } \omega_k = R \\
0 & \text{if } \omega_k = B 
\end{cases}\]

Terminal cost \( g_N(R) = g_N(B) = 0 \).
let's calculate $J_3(R)$ in detail.

$$J_3(R) = \min_{u_3 \in U_3(R)} E \left[ \text{stage cost when } \begin{cases} \omega_3 = R \\ u_3 = n \end{cases} + J_4(\omega_3) \right]$$

Using $J_4(\omega_3) = 0 \quad \forall \omega_3 \in \{R, B\}$

$$J_3(R) = \min \left\{ E \left[ \text{stage cost when } \begin{cases} \omega_3 = R \\ u_3 = n \end{cases} + J_4(\omega_3) \right] \right\}$$

$$= \min \left\{ \begin{array}{c} \frac{1}{3} \left( -100 + 0 \right) + \frac{1}{3} \left( +20 - 100 + 0 \right) \\
\frac{1}{3} \left( 0 + 0 \right) + \frac{1}{3} \left( +20 - 100 + 0 \right) \end{array} \right\}$$

$$= \min \left\{ \begin{array}{c} -30 \\
-40 \end{array} \right\} = \boxed{-40}$$
\[
\begin{align*}
  u_3 \in U_3(B) & \quad \frac{1}{\omega_3} \left[ \text{stage cost} + J_4(c_W) \right] \\
  = \{r, u\} & \\
  = \min \left\{ \begin{array}{l}
  P(\omega_3 = R \mid x_2 = B, u_3 = r) (+40 -100 + 0) + \\
  P(\omega_3 = B \mid x_3 = B, u_3 = r) (+40 + 0), \\
  P(\omega_3 = R \mid x_3 = B, u_3 = u) (+150 -100 + 0) + \\
  P(\omega_3 = B \mid x_3 = B, u_3 = u) (+150) \end{array} \right\} \\
  = \min \left\{ \begin{array}{l}
  0.6 (-60) + 0.4 (+40), \\
  1 (+50) + 0 (+150) \end{array} \right\} \\
  = \min \{ -20, +50 \} \\
\end{align*}
\]

\[J_3(B) = -20\]

\[\Rightarrow \quad u_3(B) = r\]

Similarly

week 2

\[x_2 = R \quad u_2 = n\]
\[u_2 = m\]

\[C = 0 + 0.7 (-20) + 0.3 (-100 - 40) = -56\]

\[C = 20 + 0.4 (-20) + 0.6 (-100 - 40) = -72\]

for the rest see

A graphical way

both get
-100

week 2

week 3

--- no main maintenance

\[J_3(R) = -40\]

\[J_3(B)\]