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Quiz 2 **December 8th, 2010**

Dynamic Programming & Optimal Control (151-0563-01) **Prof. R. D'Andrea**

Solutions

Duration:	**45 minutes**
Number of Problems:	**1**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 **100%**

Consider the following dynamical system,

$$\dot{x}(t) = u(t) \ , \ 0 \leq t \leq T \ , \ x(0) = x_0$$

where $x(t) \in \mathbb{R}$ and $u(t) \in \mathbb{R}$. x_0 and T are fixed and given.

a) Calculate the optimal trajectory $x^*(t)$ and optimal control input $u^*(t)$ that minimize

$$J = \frac{1}{2} \int_0^T \left(x^2(t) + u^2(t) \right) dt.$$

b) Find $x^*(t)$ and $u^*(t)$ as $T \to \infty$. Furthermore, calculate the optimal cost

$$J^*_\infty = \lim_{T\to\infty} \frac{1}{2} \int_0^T \left(x^{*2}(t) + u^{*2}(t) \right) dt.$$

c) Find a solution $V(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ to the following partial differential equation

$$0 = \min_u \left(\frac{1}{2} \left(x^2 + u^2 \right) + \frac{\partial V(t,x)}{\partial t} + \frac{\partial V(t,x)}{\partial x} u \right) , \ t \geq 0.$$

Solution 1

a) Apply the minimum principle:

- The Hamiltonian is given by

$$H(x,u,p) = \frac{1}{2}\left(x^2 + u^2\right) + pu$$

- The adjoint equations follow from the equation above

$$\dot{p}(t) = -\frac{\partial H}{\partial x} = -x(t), \qquad p(T) = 0 \text{ (no terminal cost)}$$

- The optimal input is obtained by minimizing the Hamiltonian along the optimal trajectory

$$\frac{\partial H}{\partial u} = 0 \quad \Rightarrow \quad u + p = 0 \quad \Rightarrow \quad u = -p$$

- Now $\dot{x} = -p$ and $\dot{p} = -x$ yield

$$\ddot{x} = x, \text{ with } x(0) = x_0,\ \dot{x}(T) = 0.$$

- Solving the above differential equation gives
  [**Method 1**: Candidate solution $x(t) = A\cosh(t) + B\sinh(t)$]
  Using initial conditions $x(0) = x_0$, $\dot{x}(T) = 0$ and $\dot{x} = A\sinh(t) + B\cosh(t)$,
  we get $A = x_0$ and $B = -x_0\frac{\sinh(T)}{\cosh(T)}$. This gives,

$$x(t) = x_0\cosh(t) - x_0\frac{\sinh(T)}{\cosh(T)}\sinh(t)$$

$$u(t) = \dot{x}(t) = x_0\sinh(t) - x_0\frac{\sinh(T)}{\cosh(T)}\cosh(t)$$

  [**Method 2**: Candidate solution $x(t) = A'e^t + B'e^{-t}$]
  Using initial conditions $x(0) = x_0$, $\dot{x}(T) = 0$ and $\dot{x} = A'e^t - B'e^{-t}$,
  we get $A' = \frac{x_0}{1+e^{2T}}$ and $B' = \frac{x_0}{1+e^{-2T}}$. This gives,

$$x(t) = \frac{x_0}{1+e^{2T}}e^t + \frac{x_0}{1+e^{-2T}}e^{-t}$$

$$u(t) = \dot{x}(t) = \frac{x_0}{1+e^{2T}}e^t - \frac{x_0}{1+e^{-2T}}e^{-t}$$

b) Optimal solution for infinite horizon setting:

- Using the solution of Method 1 in a)

$$x(t) = x_0\left(\frac{e^t + e^{-t}}{2} - \left(\frac{e^T - e^{-T}}{e^T + e^{-T}}\right)\left(\frac{e^t - e^{-t}}{2}\right)\right)$$

$$\text{as } T \to \infty,\ x(t) \to x_0 e^{-t}$$

$$\text{similarly, } u(t) \to -x_0 e^{-t}$$

$$J^*_\infty = \frac{1}{2}\int_0^\infty \left(x_0^2 e^{-2t} + x_0^2 e^{-2t}\right) dt = \frac{x_0^2}{2}e^{-2t}\bigg|_\infty^0 = \frac{x_0^2}{2}$$

- Note: [**Method 2**: Rigorous Proof]

$$
\begin{aligned}
J^*(t,x) &= \frac{1}{2}\int_0^T \left(x^{*2}(t) + u^{*2}(t)\right) dt. \\
&= \frac{x_0^2}{2}\frac{1-e^{-4T}}{(1+e^{-2T})^2} \\
\Rightarrow \quad J^*_\infty(t,x) &= \lim_{T\to\infty} J^*(t,x) = \frac{x_0^2}{2}
\end{aligned}
$$

c) This is the Hamilton-Jacobi-Bellman equation for the above optimal control problem. For the above problem, if we find ourselves at state x at time t, the optimal cost to go is $\frac{x^2}{2}$. Therefore, $V(t,x) = \frac{x^2}{2}$ is a candidate solution.
Verify:

$$
\frac{\partial V}{\partial t} = 0, \; \frac{\partial V}{\partial x} = x
$$

$$
\Rightarrow \min_u \left(\frac{1}{2}\left(x^2 + u^2\right) + xu\right)
$$

occurs when $u = -x$. Then we have $\frac{1}{2}\left(x^2 + x^2\right) - x^2 = 0$, as required.