ETH
Eidgenössische Technische Hochschule Zürich
Swiss Federal Institute of Technology Zurich

Institute for Dynamic Systems and Control
IDSC
Institut für Dynamische Systeme und Regelungstechnik

Quiz 1 **November 2nd, 2011**

Dynamic Programming & Optimal Control (151-0563-00) Prof. R. D'Andrea

Solutions

Duration:	**45 minutes**
Number of Problems:	**2**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 40%

Consider the system equation

$$\tilde{x}_{k+1} = \tilde{x}_k^2 + 2 \cdot u_k^2 + u_{k-2} + w_k \cdot u_k \cdot \tilde{x}_k \ , \quad k = 2, 3$$

and the cost

$$(\tilde{x}_4 - T)^2 + \sum_{k=2}^{3} \left(\tilde{x}_k^2 + u_k^2 + u_{k-2}^2 \right) ,$$

in which $T \in \mathbb{R}$ is a constant and the disturbance w_k has the following distribution

$$w_k = \begin{cases} 1 \text{ with probability } 1/4 \\ 0 \text{ with probability } 1/4 \\ -1 \text{ with probability } 1/2 \end{cases} \qquad \text{for all } k.$$

Reformulate this problem in the form of the basic problem that can directly be solved with the Dynamic Programming Algorithm, that is bring the problem to the form

$$x_{k+1} = f_k(x_k, u_k, w_k)$$

with the cost

$$g_4(x_4) + \sum_{k=2}^{3} g_k(x_k, u_k).$$

Solution 1

Using the augmented state variable

$$x_k = \begin{bmatrix} x_{k,1} \\ x_{k,2} \\ x_{k,3} \end{bmatrix} := \begin{bmatrix} \tilde{x}_k \\ u_{k-1} \\ u_{k-2} \end{bmatrix}$$

we can rewrite the system equation

$$x_{k+1} = \begin{bmatrix} x_{k,1}^2 + 2 \cdot u_k^2 + x_{k,3} + w_k \cdot u_k \cdot x_{k,1} \\ u_k \\ x_{k,2} \end{bmatrix} =: f_k(x_k, u_k, w_k),$$

the terminal cost

$$g_4(x_4) = (x_{4,1} - T)^2,$$

and the stage cost

$$g_k(x_k, u_k) = x_{k,1}^2 + x_{k,3}^2 + u_k^2, \quad k = 2, 3.$$

Problem 2 60%

Suppose we have a machine that is either running or is broken down. If it runs throughout the week, it makes a profit of $G > 0$ for that week. If it fails during the week, profit is zero for that week.

If it is running at the start of the week and we perform preventive maintenance, the probability that it will run throughout the week is p_{m}. If we do not perform such maintenance, the probability of running throughout the week is $p_{\mathrm{nm}} < p_{\mathrm{m}}$. However, maintenance will cost $C_{\mathrm{m}} > 0$.

When the machine is broken down at the start of the week, it may either be repaired at a cost of $C_{\mathrm{r}} > C_{\mathrm{m}}$, in which case it will run throughout the week with a probability of p_{m}, or it may be replaced at a cost of $C_{\mathrm{l}} > 0$ by a new machine that is guaranteed to run throughout the week.

Assume that after $N > 1$ weeks the machine, irrespective of its state, is scrapped without incurring any cost.

Recall the basic problem setup:

- System dynamics

$$\begin{aligned} x_{k+1} &= f_k\left(x_k, u_k, w_k\right), \quad k = 0, 1, \ldots, N-1 \\ \text{where } x_k &\in S_k \\ u_k &\in U_k(x_k) \\ w_k &\sim P(\cdot|x_k, u_k) \end{aligned}$$

- Cost function

$$g_N(x_N) + \sum_{k=0}^{N-1} g_k(x_k, u_k, w_k)$$

a) Problem formulation

(i) Define the state space S_k.

(ii) Define the control space $U_k(x_k)$.

(iii) Define the system dynamics in the form of $x_{k+1} = w_k$ with the conditional probability distribution $P(\cdot|x_k, u_k)$ for w_k.

(iv) Define the stage costs $g_k(x_k, u_k, w_k)$ and the terminal cost $g_N(x_N)$.

b) Applying the Dynamic Programming Algorithm

(i) Find the optimal cost-to-go at week $N-1$, $J_{N-1}(x_{N-1})$.

(ii) Derive the conditions that need to be be satisfied such that the optimal policy at week $N-1$ is to maintain a running machine and to repair a broken machine.

Solution 2

a) Problem Formulation

(i) State space $S_k = \{R, B\}$: R =machine running, B =machine broken

(ii) Control space $U_k(x_k)$:

$$U_k(R) = \{nm, m\} : nm = \text{no maintenance}, \; m = \text{maintenance}$$
$$U_k(B) = \{r, l\} : r = \text{repair}, \; l = \text{replace}$$

(iii) System dynamics: $x_{k+1} = w_k$, where the probability distribution of w_k is given by

$$\begin{aligned}
P(w_k = R \mid x_k = R, u_k = nm) &= p_{nm}, \\
P(w_k = B \mid x_k = R, u_k = nm) &= 1 - p_{nm}, \\
P(w_k = R \mid x_k = R, u_k = m) &= p_m, \\
P(w_k = B \mid x_k = R, u_k = m) &= 1 - p_m, \\
P(w_k = R \mid x_k = B, u_k = r) &= p_m, \\
P(w_k = B \mid x_k = B, u_k = r) &= 1 - p_m, \\
P(w_k = R \mid x_k = B, u_k = l) &= 1.
\end{aligned}$$

(iv) Input costs C_u:

$$\begin{aligned}
u = nm \quad &\rightarrow \quad C_{nm} = 0 \\
u = m \quad &\rightarrow \quad C_m \\
u = r \quad &\rightarrow \quad C_r \\
u = l \quad &\rightarrow \quad C_l
\end{aligned}$$

Gain (negative cost): $-G$ (If the machine ran for the full week)
Stage cost:

$$g_k = \begin{cases} C_{u_k} - G & \text{if} \quad w_k = R \; (= x_{k+1}) \\ C_{u_k} & \text{if} \quad w_k = B \; (= x_{k+1}) \end{cases}$$

Terminal cost: Since it is assumed that after N weeks the machine, irrespective of its state, is scrapped without incurring any cost gives

$$g_{\mathrm{N}}(R) = 0,$$
$$g_{\mathrm{N}}(B) = 0.$$

b) Applying the Dynamic Programming Algorithm (DPA)

(i) Applying the Dynamic Programming Algorithm for week N-1:

$$J_{\mathrm{N}}(R) = g_{\mathrm{N}}(R) = 0,$$
$$J_{\mathrm{N}}(B) = g_{\mathrm{N}}(B) = 0.$$

$$\begin{aligned}
J_{N-1}(R) &= \min_{u_{N-1}} \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N)\} \\
&= \min \left(\mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = m\}, \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = nm\} \right) \\
&= \min \left(C_m + p_m(-G), p_{nm}(-G) \right). \qquad (1)
\end{aligned}$$

$$\begin{aligned}
J_{N-1}(B) &= \min_{u_{N-1}} \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N)\} \\
&= \min\left(\mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = r\}, \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = l\} \right) \\
&= \min\left(C_r + p_m(-G), C_l - G\right). \qquad (2)
\end{aligned}$$

(ii) Finding the conditions for optimal policies:

$$\begin{aligned}
& \mu_{N-1}(R) = m \quad \text{[The optimal policy for week } N-1 \text{ is to maintain a running machine.]} \\
\overset{(1),\,\mathrm{DPA}}{\Leftrightarrow} \quad & \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = m\} - \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = nm\} < 0 \\
\overset{(1)}{\Leftrightarrow} \quad & C_m + p_m(-G) - p_{nm}(-G) = \underline{C_m - (p_m - p_{nm})G < 0.}
\end{aligned}$$

$$\begin{aligned}
& \mu_{N-1}(B) = r \quad \text{[The optimal policy for week } N-1 \text{ is to repair a broken machine.]} \\
\overset{(2),\,\mathrm{DPA}}{\Leftrightarrow} \quad & \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = r\} - \mathop{\mathrm{E}}_{w_{N-1}} \{g_{N-1} + J_N(x_N) \mid u_{N-1} = l\} < 0 \\
\overset{(2)}{\Leftrightarrow} \quad & C_r + p_m(-G) - (C_l - G) = \underline{C_r - C_l + (1 - p_m)G < 0.}
\end{aligned}$$