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Quiz 1 **October 14th, 2009**

Dynamic Programming & Optimal Control (151-0563-00)

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Solutions

Duration:	**45 minutes**
Number of Problems:	**3**
Permitted Aids:	None. Use only the prepared sheets for your solutions.

Problem 1 25%

Consider the system equation

$$\tilde{x}_{k+1} = \tilde{f}_k\big(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k\big),$$

and the cost

$$\tilde{g}_N\big(\tilde{x}_N\big) + \sum_{k=0}^{N-1} \tilde{g}_k\big(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k\big).$$

Reformulate this problem in the form of the basic problem that can directly be solved with the Dynamic Programming Algorithm, that is bring the problem in the form

$$x_{k+1} = f_k\big(x_k, u_k\big),$$

with the cost

$$g_N\big(x_N\big) + \sum_{k=0}^{N-1} g_k\big(x_k, u_k\big).$$

Solution 1

Introducing the new state variable

$$x_k = \begin{bmatrix} x_{1,k} \\ x_{2,k} \\ x_{3,k} \end{bmatrix} := \begin{bmatrix} \tilde{x}_k \\ \tilde{x}_{k-1} \\ \tilde{x}_{k-2} \end{bmatrix},$$

and the input variable $u_k := \tilde{u}_k$, we can rewrite the system equation

$$x_{k+1} := \begin{bmatrix} \tilde{x}_{k+1} \\ \tilde{x}_k \\ \tilde{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{f}_k\big(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k\big) \\ \tilde{x}_k \\ \tilde{x}_{k-1} \end{bmatrix} = \begin{bmatrix} \tilde{f}_k\big(x_{1,k}, x_{3,k}, u_k\big) \\ x_{1,k} \\ x_{2,k} \end{bmatrix} =: f_k\big(x_k, u_k\big). \tag{1}$$

The cost becomes

$$\begin{aligned} \tilde{g}_N\big(\tilde{x}_N\big) + \sum_{k=0}^{N-1} \tilde{g}_k\big(\tilde{x}_k, \tilde{x}_{k-2}, \tilde{u}_k\big) &= \tilde{g}_N\big(x_{N,1}\big) + \sum_{k=0}^{N-1} \tilde{g}_k\big(x_{1,k}, x_{3,k}, u_k\big) \\ &= g_N\big(x_N\big) + \sum_{k=0}^{N-1} g_k\big(x_k, u_k\big), \end{aligned} \tag{2}$$

where we defined $g_N\big(x_N\big) := \tilde{g}_N\big(x_{1,N}\big)$ and $g_k\big(x_k, u_k\big) := \tilde{g}_k\big(x_{1,k}, x_{3,k}, u_k\big)$. The system equation (1) and the cost (2) are in the desired form of the basic problem.

Problem 2 25%

Consider the cost g_k at stage k:

$$g_k(x_k, u_k, w_k) = x_k^2 + 2x_k u_k w_k + 2w_k^2,$$

and the conditional probability distribution for w_k

$$\begin{aligned} P(w_k = 0|x_k = 0) &= \tfrac{1}{2} \\ P(w_k = 1|x_k = 0) &= \tfrac{1}{2} \end{aligned} \qquad\qquad \begin{aligned} P(w_k = -1|x_k = 1) &= \tfrac{1}{6} \\ P(w_k = 0|x_k = 1) &= \tfrac{1}{2} \\ P(w_k = 1|x_k = 1) &= \tfrac{1}{3}. \end{aligned}$$

Given the state $x_k = 1$ and the input $u_k = 2$, compute the expected value

$$\underset{w_k}{E}\big(g_k(x_k, u_k, w_k)\big).$$

Solution 2

In general, the expected value of a function $f(\cdot)$ of a discrete random variable W that takes the discrete values $w_1, w_2, \ldots, w_N$ with probability distribution $P(w_i|x_k)$ conditioned on x_k is

$$E\big(f(W)\big) = \sum_{i=1}^{N} f(w_i)P(w_i|x_k).$$

For the given problem, since $x_k = 1$ is given, the expected value of g_k is

$$\begin{aligned}
E\big(g_k(x_k, u_k, w_k)\big) &= \sum_{i=1}^{3} g_k(x_k = 1, u_k = 2, w_i)P(w_i|x_k = 1) \\
&= \frac{1}{6}g_k(1,2,-1) + \frac{1}{2}g_k(1,2,0) + \frac{1}{3}g_k(1,2,1) \\
&= \frac{1}{6}\cdot\big(1^2 + 2\cdot 1\cdot 2\cdot(-1) + 2\cdot(-1)^2\big) \\
&\quad + \frac{1}{2}\cdot\big(1^2 + 2\cdot 1\cdot 2\cdot 0 + 2\cdot 0^2\big) \\
&\quad + \frac{1}{3}\cdot\big(1^2 + 2\cdot 1\cdot 2\cdot 1 + 2\cdot 1^2\big) \\
&= -\frac{1}{6} + \frac{1}{2} + \frac{7}{3} \\
&= \frac{8}{3}.
\end{aligned}$$

Problem 3 50%

Consider the dynamic system

$$x_{k+1} = x_k + u_k, \qquad x_k,\, u_k \in \mathbb{R}\,,$$

with initial state x_0. The cost function to be minimized is given by

$$x_2^2 \; + \; u_0^2 \; + \; u_1^2\,.$$

Apply the Dynamic Programming algorithm to find the optimal control policy $u_k^* = \mu_k^*\left(x_k\right),\;\; k = 0, 1$, for the following two cases:

a) no constraints on u_k.

b) u_k can only take the values 1 and -1.

Solution 3

The optimal control problem is considered over a time horizon $N = 2$ and the cost, to be minimized, is defined by

$$g_2(x_2) = x_2^2 \qquad \text{and} \qquad g_k(x_k, u_k, w_k) = u_k^2, \quad k = 0, 1.$$

a) With $u_k \in \mathbb{R}$ and no further constraints, the DP algorithm proceeds as follows:

2nd stage:

$$J_2(x_2) = x_2^2$$

1st stage:

$$J_1(x_1) = \min_{u_1} \left[u_1^2 + J_2(x_2)\right] = \min_{u_1} \underbrace{\left[u_1^2 + (x_1 + u_1)^2\right]}_{L_1(x_1,u_1)}$$

Since u_1 is continuous, the minimizing input is found by

$$\frac{\partial L_1}{\partial u_1} = 2u_1 + 2\,(x_1 + u_1) \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad u_1 = -\frac{1}{2}x_1\,.$$

The sufficient condition $\frac{\partial^2 L_1}{\partial u_1^2} > 0$ is satisfied. Therefore,

$$\Rightarrow \; \mu_1^*(x_1) = u_1^* = -\frac{1}{2}x_1 \qquad \forall\, x_1 \in \mathbb{R}\,,$$

$$\Rightarrow \; J_1(x_1) = \frac{1}{2}x_1^2\,.$$

0th stage:

$$J_0(x_0) = \min_{u_0} \left[u_0^2 + J_1(x_1)\right] = \min_{u_0} \underbrace{\left[u_0^2 + \frac{1}{2}\,(x_0 + u_0)^2\right]}_{L_0(x_0,u_0)}$$

Again, since u_0 is continuous, the minimizing input is found by

$$\frac{\partial L_0}{\partial u_0} = 2u_0 + (x_0 + u_0) \stackrel{!}{=} 0 \qquad \Leftrightarrow \qquad u_0 = -\frac{1}{3}x_0\,.$$

The sufficient condition $\frac{\partial^2 L_0}{\partial u_0^2} > 0$ is satisfied. Therefore,

$$\Rightarrow \; \mu_0^*(x_0) = u_0^* = -\frac{1}{3}x_0 \qquad \forall\, x_0 \in \mathbb{R}\,,$$

$$\Rightarrow \; J_0(x_0) = \frac{1}{3}x_0^2\,.$$

b) For the constrained set $u_k \in \{-1, 1\}$, the DP algorithm results in:

2nd stage:

$$J_2(x_2) = x_2^2$$

1st stage:

$$\begin{aligned} J_1(x_1) &= \min_{u_1} \left[u_1^2 + (x_1 + u_1)^2\right] = 1 + \min_{u_1} \left[(x_1 + u_1)^2\right] \\ &= 2 + x_1^2 + 2 \cdot \min_{u_1} \left[x_1 u_1\right] \end{aligned}$$

Depending on the sign of x_1,

$$\operatorname{sgn}(x_1) = \begin{cases} 1 & \text{for} \quad x_1 \geq 0 \\ -1 & \text{for} \quad x_1 < 0\,, \end{cases}$$

the minimizing input u_1^* is found as

$$\begin{aligned} &\Rightarrow \ \mu_1^*(x_1) = u_1^* = -\operatorname{sgn}(x_1) \qquad \forall\, x_1 \in \mathbb{R}\,, \\ &\Rightarrow \ J_1(x_1) = 1 + \big(1 - |x_1|\big)^2 = 2 + x_1^2 - 2\,|x_1|\ . \end{aligned}$$

0th stage:

$$\begin{aligned} J_0(x_0) &= \min_{u_0} \left[u_0^2 + J_1(x_1)\right] = \min_{u_0} \left[u_0^2 + 1 + (1 - |x_0 + u_0|)^2\right] \\ &= 2 + \min_{u_0} \left[(1 - |x_0 + u_0|)^2\right] \\ &= 4 + x_0^2 + 2 \cdot \min_{u_0} \left[x_0 u_0 - \big|x_0 + u_0\big|\right] \end{aligned}$$

Distinguishing the cases $x_0 < -1$, $-1 \leq x_0 \leq 1$ and $x_0 > 1$, yields

$$\mu_0^*(x_0) = u_0^* = \begin{cases} 1 & \text{for} \quad x_0 < -1 \\ \pm 1 & \text{for} \quad -1 \leq x_0 \leq 1 \\ -1 & \text{for} \quad x_0 > 1\,. \end{cases}$$

In short, *one* optimal solution is

$$\Rightarrow \ \mu_0^*(x_0) = u_0^* = -\operatorname{sgn}(x_0) \qquad \forall\, x_0 \in \mathbb{R}\,,$$

$$\Rightarrow \ J_0(x_0) = \begin{cases} 2 + x_0^2 & \text{for} \quad -1 \leq x_0 \leq 1 \\ 6 + x_0^2 - 4\,|x_0| & \text{otherwise}\,. \end{cases}$$